

LOCAL MATRIX HOMOTOPIES AND SOFT TORI

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ABSTRACT. We present solutions to local connectivity problems in matrix representations of the form $C([-1, 1]^N) \rightarrow A_{n,\varepsilon} \leftarrow C_\varepsilon(\mathbb{T}^2)$ for any $\varepsilon \in [0, 2]$ and any integer $n \geq 1$, where $A_{n,\varepsilon} \subseteq M_n$ and where $C_\varepsilon(\mathbb{T}^2)$ denotes the **Soft Torus**. We solve the connectivity problems by introducing the so called toroidal matrix links, which can be interpreted as normal contractive matrix analogies of free homotopies in differential algebraic topology.

In order to deal with the locality constraints, we have combined some techniques introduced in this document with some techniques from matrix geometry, combinatorial optimization, classification and representation theory of C^* -algebras.

1. INTRODUCTION

In this document we study the solvability of some local connectivity problems via constrained normal matrix homotopies in C^* -representations of the form

$$(1.1) \quad C(\mathbb{T}^N) \longrightarrow M_n,$$

for a fixed but arbitrary integer $N \geq 1$ and any integer $n \geq 1$. In particular we study local normal matrix homotopies which preserve commutativity and also satisfy some additional constraints, like being rectifiable or piecewise analytic.

We build on some homotopic techniques introduced initially by Bratteli, Elliot, Evans and Kishimoto in [3] and generalized by Lin in [18] and [22]. We combine the homotopic techniques with some techniques introduced here and some other techniques from matrix geometry and noncommutative topology developed by Loring [24, 27], Shulman [27], Bhatia [1], Chu [7], Brockett [4], Choi [6, 5], Effros [5], Exel [10], Eilers [10], Elsner [11], Pryde [30, 29], McIntosh [29] and Ricker [29], to construct the so called *toroidal matrix links*, which we use to obtain the main theorems presented in section §4, and which consist on local connectivity results in matrix representations of the form 1.1 and also of the form

$$(1.2) \quad C(\mathbb{T}^N) \longrightarrow M_n \longleftarrow C([-1, 1]^N).$$

Toroidal matrix links can be interpreted as noncommutative analogies of free homotopies in algebraic topology and topological deformation theory, they are introduced in section §3 together with some other matrix geometrical objects.

In §4.3 we present a connectivity technique which provides us with very important information on the local uniform connectivity in matrix representations of the form $C(\mathbb{T}^2) \rightarrow M_n$.

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Given $\delta > 0$, a function $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}_0^+$ and two matrices x, y in a set $S \subseteq M_n$ such that $\|x - y\| \leq \delta$, by a $\varepsilon(\delta)$ -**local matrix homotopy** between x and y , we mean a matrix path $X \in C([0, 1], M_n)$ such that $X_0 = x$, $X_1 = y$, $X_t \in S$ and $\|X_t - y\| \leq \varepsilon(\delta)$ for each $t \in [0, 1]$. We write $x \rightsquigarrow_\varepsilon y$ to denote that there is a ε -local matrix homotopy between x and y .

The motivation and inspiration to study local normal matrix homotopies which preserve commutativity in C^* -representations of the form 1.1 and 1.2, came from mathematical physics [15, §3] and matrix approximation theory [8].

The problems from mathematical physics which motivated this study are inverse spectral problems, which consist on finding for a certain set of matrices X_1, \dots, X_N which *approximately satisfy* a set of polynomial constraints $\mathcal{R}(x_1, \dots, x_N)$ on N NC-variables, a set of *nearby matrices* $\tilde{X}_1, \dots, \tilde{X}_N$ which approximate X_1, \dots, X_N and exactly satisfy the constraints $\mathcal{R}(x_1, \dots, x_N)$. The problems from matrix approximation theory that we considered for this study, are of the type that can be reduced to the study of the solvability conditions for approximate and exact joint diagonalization problems for N -tuples of normal matrix contractions.

Since the problems which motivated the research reported in this document can be restated in terms of the study local piecewise analytic connectivity in matrix representations of the form $C_\varepsilon(\mathbb{T}^2) \rightarrow M_n \leftarrow C(\mathbb{T}^N)$ and $C_\varepsilon(\mathbb{T}^2) \rightarrow M_n \leftarrow C([-1, 1]^N)$, we studied several variations of problems of the form.

Problem 1 (Lifted connectivity problem). *Given $\varepsilon > 0$, is there $\delta > 0$ such that the following conditions hold? For any integer $n \geq 1$, some prescribed sequence of linear compressions $\kappa_n : M_{mn} \rightarrow M_n$ for some $m \geq 1$, and any two families of N pairwise commuting normal contractions X_1, \dots, X_N and Y_1, \dots, Y_N in M_n which satisfy the constraints $\|X_j - Y_j\| \leq \delta$, $1 \leq j \leq N$, there are two families of N pairwise commuting normal contractions $\tilde{X}_1, \dots, \tilde{X}_N$ and $\tilde{Y}_1, \dots, \tilde{Y}_N$ in M_{mn} which satisfy the relations: $\kappa(\tilde{X}_j) = X_j$, $\kappa(\tilde{Y}_j) = Y_j$ and $\|\tilde{X}_j - \tilde{Y}_j\| \leq \varepsilon$, $1 \leq j \leq N$. Moreover, there are N peicewise analytic ε -local homotopies of normal contractions $\mathbf{X}^1, \dots, \mathbf{X}^N \in C([0, 1], M_{mn})$ between the corresponding pairs \tilde{X}_j, \tilde{Y}_j in M_{mn} , which satisfy the relations $\mathbf{X}_t^j \mathbf{X}_t^k = \mathbf{X}_t^k \mathbf{X}_t^j$, for each $1 \leq j, k \leq N$ and each $0 \leq t \leq 1$.*

By solving problem **P.1** we learned about the local connectivity of arbitrary δ -close N -tuples of pairwise commuting normal contractions X_1, \dots, X_N and Y_1, \dots, Y_N in M_n , which was the main motivation of the research reported here. We also obtained some results concerning to the geometric structure of the joint spectra (in the sense of [29]) of the N -tuples.

For a given $\delta > 0$, the study of the solvability conditions of problems of the form **P.1** provided us with geometric information about local deformations of particular representations of the form $C(\mathbb{T}^N) \rightarrow A_0 := C^*(U_1, \dots, U_N) \subseteq M_n$ and $C(\mathbb{T}^N) \rightarrow A_1 := C^*(V_1, \dots, V_N) \subseteq M_n$, where $U_1, \dots, U_N, V_1, \dots, V_N \in \mathcal{U}(n)$ are pairwise commuting unitary matrices such that $\|U_j - V_j\| \leq \delta$. By local deformations we mean a family $\{A_t\}_{t \in [0, 1]} \subseteq M_n$ of abelian C^* -algebras, with $A_t := C^*(\mathbf{X}_t^1, \dots, \mathbf{X}_t^N)$ and where $\mathbf{X}_t^1, \dots, \mathbf{X}_t^N \in C([0, 1], \mathcal{U}(n))$ are $\varepsilon(\delta)$ -local matrix homotopies between U_1, \dots, U_N and V_1, \dots, V_N for some function $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}_0^+$.

The main results are presented in §4, in section §4.2 we use toroidal matrix links to obtain some local piecewise analytic connectivity results which are non-uniform in dimension. In section §4.2.1 we derive an uniform approximate connectivity

technique via matrix homotopy lifting and in section §4.4 we present a connectivity lemma that can be used to derive some uniform connectivity results between matrix representations of finite sets of universal algebraic contractions, some of the details of these constructions will be presented in [33].

2. PRELIMINARIES AND NOTATION

2.1. Matrix Sets and Operations. Given two elements x, y in a C^* -algebra A , we will write $[x, y]$ and $\text{Ad}[x](y)$ to denote the operations $[x, y] := xy - yx$ and $\text{Ad}[x](y) := xyx^*$.

Given any C^* -algebra A and any element x in $M_n(A)$, we will denote by $\text{diag}_n[x]$ the operation defined by the expression

$$\begin{aligned} M_n(A) &\rightarrow M_n(A) \\ x &\mapsto \text{diag}_n[x] \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} &\mapsto \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ 0 & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nn} \end{pmatrix}. \end{aligned}$$

Given a C^* -algebra A , we write $\mathcal{N}(A)$, $\mathbb{H}(A)$ and $\mathbb{U}(A)$ to denote the sets of normal, hermitian and unitary elements in A respectively. We will write $\mathcal{N}(n)$, $\mathbb{H}(n)$ and $\mathbb{U}(n)$ instead of $\mathcal{N}(M_n)$, $\mathbb{H}(M_n)$ and $\mathbb{U}(M_n)$. A normal element u in a C^* -algebra A is called a partial unitary if the element $uu^* = p$ is an orthogonal projection in A , i.e. p satisfies the relations $p = p^* = p^2$, we denote by $\mathbb{PU}(A)$ the set of partial unitaries in A and we write $\mathbb{PU}(n)$ instead of $\mathbb{PU}(M_n)$.

We write \mathbb{I} , \mathbb{J} , \mathbb{T}^1 and \mathbb{D}^2 to denote the sets $\mathbb{I} := [0, 1]$, $\mathbb{J} = [-1, 1]$, $\mathbb{T}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ and $\mathbb{D}^2 := \{z \in \mathbb{C} \mid |z| \leq 1\}$. For some arbitrary matrix set $S \subseteq M_n$ and some arbitrary compact set $\mathbb{X} \subset \mathbb{C}$, we will write $S(\mathbb{X})$ to denote the subset of elements in S described by the expression,

$$(2.1) \quad S(\mathbb{X}) := \{x \in S \mid \sigma(x) \subseteq \mathbb{X}\},$$

for instance we can write $\mathcal{N}(n)(\mathbb{D}^2)$ to denote the set of normal contractions. We will denote by \mathcal{M}_∞ the C^* -algebra described by

$$(2.2) \quad \mathcal{M}_\infty := \overline{\bigcup_{n \in \mathbb{Z}^+} M_n}^{\|\cdot\|}.$$

In this document we write $\mathbb{1}_n$ to denote the identity matrix in M_n . The symbol \mathbf{N}_n will be used to denote the diagonal matrices

$$(2.3) \quad \mathbf{N}_n := \text{diag}[n, n-1, \dots, 2, 1].$$

We will write Ω_n and Σ_n to denote the unitary matrices defined by

$$(2.4) \quad \Omega_n := e^{\frac{2\pi i}{n} \mathbf{N}_n} = \text{diag}\left[1, e^{\frac{2\pi i(n-1)}{n}}, \dots, e^{\frac{4\pi i}{n}}, e^{\frac{2\pi i}{n}}\right]$$

and

$$(2.5) \quad \Sigma_n := \begin{pmatrix} 0 & \mathbb{1}_{n-1} \\ 1 & 0 \end{pmatrix}.$$

Remark 2.1. The unitary matrices Ω_n and Σ_n are related, by the equation

$$\Omega_n = \mathcal{F}_n^* \Sigma_n \mathcal{F}_n,$$

where $\mathcal{F}_N := \left(\frac{1}{\sqrt{N}} e^{\frac{2\pi i(j-1)(k-1)}{N}} \right)_{1 \leq j, k \leq N}$ is the discrete Fourier transform (DFT) unitary matrix.

Given an abstract object (group or C^* -algebra) A we write A^{*N} to denote the operation consisting on taking the free product of N copies of A .

Definition 2.1 (Local preservers). *Given a linear mapping $K : M_N \rightarrow M_n$, with $n \leq N$, and given a set $S \subseteq M_n$, we say that K **locally preserves** S if there is $T \subseteq M_N$ such that $K(T) \subseteq S$, if in particular $K(T) \subseteq \mathcal{N}(n)$ we say that K **locally preserves normality**.*

Example 2.1. The linear compression $\kappa : M_{2n} \rightarrow M_n$ defined by

$$\kappa : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x_{11}$$

locally preserves normality with respect to the set $T := \{X \in M_{2n} | x_{11} \in \mathcal{N}(n)\}$.

Example 2.2. The linear map $\phi : M_n \rightarrow M_n, x \mapsto \mathbf{D}x$ with $n \geq 1$ and $\mathbf{D} = \frac{1}{n} \text{diag}[1, \dots, n]$, locally preserves commutativity with respect to the set $C^*(\mathbf{D})$.

2.2. Joint Spectral Variation.

2.2.1. Clifford Operators. Using the same notation as Pryde in [30], let $\mathbb{R}_{(N)}$ denote the Clifford algebra over \mathbb{R} with generators e_1, \dots, e_N and relations $e_i e_j = -e_j e_i$ for $i \neq j$ and $e_i^2 = -1$. Then $\mathbb{R}_{(N)}$ is an associative algebra of dimension 2^N . Let $S(N)$ denote the set $\mathcal{P}(\{1, \dots, N\})$. Then the elements $e_S = e_{s_1} \cdots e_{s_k}$ form a basis when $S = \{s_1, \dots, s_k\}$ and $1 \leq s_1 < \dots < s_k \leq N$. Elements of $\mathbb{R}_{(N)}$ are denoted by $\lambda = \sum_S \lambda_S e_S$ where $\lambda_S \in \mathbb{R}$. Under the inner product $\langle \mu, \lambda \rangle = \sum_S \lambda_S \mu_S$, $\mathbb{R}_{(N)}$ becomes a Hilbert space with orthonormal basis $\{e_S\}$.

The *Clifford operator* of N elements $X_1, \dots, X_N \in M_n$ is the operator defined in $M_n \otimes \mathbb{R}_{(N)}$ by

$$\text{Cliff}(X_1, \dots, X_N) := i \sum_{j=1}^N X_j \otimes e_j.$$

Each element $T = \sum_S T_S \otimes e_S \in M_n \otimes \mathbb{R}_{(N)}$ acts on elements $x = \sum_S x_S \otimes e_S \in \mathbb{C}^n \otimes \mathbb{R}_{(N)}$ by $T(x) := \sum_{S, S'} T_S(x_{S'}) \otimes e_S e_{S'}$. So $\text{Cliff}(X_1, \dots, X_N) \in M_n \otimes \mathbb{R}_{(N)} \subseteq \mathcal{L}(\mathbb{C}^n \otimes \mathbb{R}_{(N)})$. By $\|\text{Cliff}(X_1, \dots, X_N)\|$ we will mean the operator norm of $\text{Cliff}(X_1, \dots, X_N)$ as an element of $\mathcal{L}(\mathbb{C}^n \otimes \mathbb{R}_{(N)})$. As observed by Elsner in [11, 5.2] we have that

$$(2.6) \quad \|\text{Cliff}(X_1, \dots, X_N)\| \leq \sum_{j=1}^N \|X_j\|.$$

2.2.2. Joint Spectral Matchings. It is often convenient to have N -tuples (or $2N$ -tuples) of matrices with real spectra. For this purpose we use the following construction, initiated by McIntosh and Pryde. If $X = (X_1, \dots, X_N)$ is a N -tuple of n by n matrices then we can always decompose X_j in the form $X_j = X_{1j} + iX_{2j}$ where the X_{kj} all have real spectra. We write $\pi(X) := (X_{11}, \dots, X_{1N}, X_{21}, \dots, X_{2N})$ and call $\pi(X)$ a partition of X . If the X_{kj} all commute we say that $\pi(X)$ is a commuting

partition, and if the X_{kj} are simultaneously triangularizable $\pi(X)$ is a triangularizable partition. If the X_{kj} are all semisimple (diagonalizable) then $\pi(X)$ is called a semisimple partition.

We say that N normal matrices $X_1, \dots, X_N \in M_n$ are *simultaneously diagonalizable* if there is a unitary matrix $Q \in M_n$ such that $Q^* X_j Q$ is diagonal for each $j = 1, \dots, N$. In this case, for $1 \leq k \leq n$, let $\Lambda^{(k)}(X_j) := (Q^* X_j Q)_{kk}$ the (k, k) element of $Q^* X_j Q$, and set $\Lambda^{(k)}(X_1, \dots, X_N) := (\Lambda^{(k)}(X_1), \dots, \Lambda^{(k)}(X_N)) \in \mathbb{C}^N$. The set

$$\Lambda(X_1, \dots, X_N) := \{\Lambda^{(k)}(X_1, \dots, X_N)\}_{1 \leq k \leq n}$$

is called the joint spectrum of X_1, \dots, X_N . We will write $\Lambda(X_j)$ to denote the j -component of $\Lambda(X_1, \dots, X_N)$, in other words we will have that

$$\Lambda(X_j) = \text{diag} \left[\Lambda^{(1)}(X_j), \dots, \Lambda^{(N)}(X_j) \right].$$

The following theorem was proved in McIntosh, Pryde and Ricker [29].

Theorem 2.1 (McIntosh, Pryde and Ricker). *Let $X = (X_1, \dots, X_N)$ and $Y = (Y_1, \dots, Y_N)$ be N -tuples of commuting n by n normal matrices. There exists a permutation τ of the index set $\{1, \dots, n\}$ such that*

$$(2.7) \quad \|\Lambda^{(k)}(X_1, \dots, X_N) - \Lambda^{(\tau(k))}(Y_1, \dots, Y_N)\| \leq e_{N,0} \|\text{Cliff}(X_1 - Y_1, \dots, X_N - Y_N)\|$$

for all $k \in \{1, \dots, n\}$.

In this theorem, $e_{N,0}$ is an explicit constant depending only on N defined in [29, (2.4)].

2.3. Amenable C^* -algebras and Bott elements. The following lemma has been proved by H. Lin in [19].

Lemma 2.1 (H. Lin.). *For any $\varepsilon > 0$ and $d > 0$, there exists $\delta > 0$ satisfying the following: Suppose that A is a unital C^* -algebra and $u \in A$ is a unitary such that $\mathbb{T}^1 \setminus \sigma(u)$ contains an arc with length d . Suppose that $a \in A$ with $\|a\| \leq 1$ such that*

$$\|ua - au\| < \delta.$$

Then there is a self-adjoint element $h \in A$ such that $u = e^{ih}$,

$$\|ha - ah\| < \varepsilon \quad \text{and} \quad \|e^{ith}a - ae^{ith}\| < \varepsilon$$

for all $t \in \mathbb{I}$. If, furthermore, $a = p$ is a projection, we have

$$\left\| pup - p + \sum_{n=1}^{\infty} \frac{(iphp)^n}{n!} \right\| < \varepsilon.$$

The following lemma was proved by H. Lin in [22] using L.2.1, since for any integer $n \geq 1$ and any $u \in \mathbb{U}(n)$, we will have that $\mathbb{T}^1 \setminus \sigma(u)$ contains an arc with length at least $2\pi/n$.

Lemma 2.2 (H. Lin.). *Let $\varepsilon > 0$, $n \geq 1$ be an integer and $M > 0$. There exists $\delta > 0$ satisfying the following: For any finite set $\mathcal{F} \subset M_n$ with $\|a\| \leq M$ for all $a \in \mathcal{F}$, and a unitary $u \in M_n$ such that*

$$\|ua - au\| < \delta \quad \text{for all } a \in \mathcal{F},$$

there exists a continuous path of unitaries $\{u(t)\}_{t \in \mathbb{I}} \subset M_n$ with $u(0) = u$ and $u(1) = \mathbb{1}_n$ such that

$$\|u(t)a - au(t)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

Moreover,

$$\text{Length}(\{u(t)\}) \leq 2\pi.$$

Definition 2.2. (The obstruction $Bott(u, v)$.) Given two unitaries in a K_1 -simple real rank zero C^* -algebra A that almost commute, the obstruction $Bott(u, v)$ is the Bott element associated to the two unitaries as defined by Loring in [24]. It is defined whenever $\|uv - vu\| \leq \nu_0$, where ν_0 is a universal constant. It is defined as the K_0 -class

$$Bott(u, v) = [\chi_{[1/2, \infty)}(e(u, v))] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right],$$

where $e(u, v)$ is a self-adjoint element of $M_2(A)$ of the form

$$e(u, v) = \begin{pmatrix} f(v) & h(v)u + g(v) \\ u^*h(v) + g(v) & 1 - f(v) \end{pmatrix},$$

where f, g, h are certain universal real-valued continuous functions on \mathbb{T}^1 .

For details on the subject of K -theory for C^* -algebras the reader is referred to [31]. As observed by Bratteli, Elliot, Evans and Kishimoto in [3], given a pair $u, v \in \mathcal{U}(A)$ we have that the obstruction $Bott(u, v)$ needs to vanish in order to be able to solve the problem $uvu^* \rightsquigarrow_{\varepsilon(\delta)} v$ by deforming $u \in \mathcal{U}_0(A)$ to 1 continuously in $\mathcal{U}(A)$, when $\|uv - vu\| \leq \delta$.

3. MATRIX VARIETIES AND TOROIDAL MATRIX LINKS

Let us denote by \mathcal{H} a universal separable Hilbert space, by $\mathbb{B}(\mathcal{H})$ the C^* -algebra of bounded operators on \mathcal{H} , and for any given $S \subseteq \mathbb{B}(\mathcal{H})$ let us denote by $\mathbf{B}_r(S)$ the closed r -ball in S defined by $\mathbf{B}_r(S) := \{x \in S \mid \|x\| \leq r\}$.

Given some $N \in \mathbb{Z}^+$ and a set $\mathcal{R}(S) = \mathcal{R}(y_1, \dots, y_N)$ of normed polynomial relations on the N -set $S := \{y_1, \dots, y_N\}$ of NC-variables, we will call the set $\mathcal{Z}[\mathcal{R}]$ described by

$$(3.1) \quad \mathcal{Z}[\mathcal{R}] := \{x_1, \dots, x_N \mid \mathcal{R}(x_1, \dots, x_N)\}$$

with $x_1, \dots, x_N \in \mathbf{B}_1(\mathbb{B}(\mathcal{H}))$, a noncommutative semialgebraic set.

Example 3.1. As an example of normed NC-polynomial relations we can consider the set $\mathcal{R}(x, y) := \{\|x^4 - 1\| \leq 10^{-10}, \|y^7 - 1\| \leq 10^{-10}, \|xy - yx\| \leq \frac{1}{8}, xx^* = x^*x = 1, yy^* = y^*y = 1\}$.

Given a NC-semialgebraic set $\mathcal{Z}[\mathcal{R}]$, we will use the symbol $\mathcal{EZ}[\mathcal{R}]$ to denote the universal C^* -algebra

$$(3.2) \quad \mathcal{EZ}[\mathcal{R}] := C^* \langle x_1, \dots, x_N \mid \mathcal{R}(x_1, \dots, x_N) \rangle,$$

which we call the environment C^* -algebra of $\mathcal{Z}[\mathcal{R}]$. For details on universal C^* -algebras described in terms of generators and relations the reader is referred to [26].

Definition 3.1 (Semialgebraic Matrix Varieties). *Given $J \in \mathbb{Z}^+$, a system of J polynomials $p_1, \dots, p_J \in \Pi_{\langle N \rangle} = \mathbb{C} \langle x_1, \dots, x_N \rangle$ in N NC-variables $x_1, \dots, x_N \in \Pi_{\langle N \rangle}$ and a real number $\varepsilon \geq 0$, a particular matrix representation of the noncommutative semialgebraic set $\mathcal{Z}_{\varepsilon, n}(p_1, \dots, p_J)$ described by*

$$(3.3) \quad \mathcal{Z}_{\varepsilon, n}(p_1, \dots, p_J) := \{ X_1, \dots, X_N \in M_n \mid \|p_j(X_1, \dots, X_N)\| \leq \varepsilon, 1 \leq j \leq J \},$$

*will be called a ε, n -semialgebraic matrix variety (ε, n -SMV), if $\varepsilon = 0$ we can refer to the set as a **matrix variety**.*

Example 3.2. *As a first example, we will have that the set $\mathbf{Z}_n := \{X \in M_n \mid \mathbf{N}_n X - X \mathbf{N}_n = 0\}$ is a matrix variety defined by the set with one NC-polynomial relation $\{\mathbf{N}_n X - X \mathbf{N}_n = 0\}$. If for some $\delta > 0$, we set now $\mathbf{Z}_{n, \delta} := \{X \in M_n \mid \|\mathbf{N}_n X - X \mathbf{N}_n\| \leq \delta\}$, the set $\mathbf{Z}_{n, \delta}$ is a matrix semialgebraic variety defined by the set with one normed NC-polynomial relation $\{\|\mathbf{N}_n X - X \mathbf{N}_n\| \leq \delta\}$.*

Example 3.3. *Other example of a matrix semialgebraic variety, that has been useful to understand the geometric nature of the problems solved in this document, is described by the matrix set $\mathbf{Iso}_\delta(x, y)$, defined for some given $\delta \geq 0$ and any two normal contractions x and y in M_n , by the expression*

$$\mathbf{Iso}_\delta(x, y) := \left\{ (z, w) \in \mathcal{N}(n)(\mathbb{D}^2) \times \mathbb{U}(n) \mid \begin{array}{l} \|xw - wz\| = 0 \\ \|[z, y]\| = 0 \\ \|z - y\| \leq \delta \end{array} \right\}.$$

3.1. Toroidal Matrix Links.

3.1.1. Finsler manifolds, matrix paths and toroidal matrix links.

Definition 3.2 (Finsler manifold). *A Finsler manifold is a pair (M, F) where M is a manifold and $F : TM \rightarrow [0, \infty)$ is a function (called a Finsler norm) such that*

- *F is smooth on $TM \setminus \{0\} = \bigcup_{x \in M} \{T_x M \setminus \{0\}\}$,*
- *$F(v) \geq 0$ with equality if and only if $v = 0$,*
- *$F(\lambda v) = \lambda F(v)$ for all $\lambda \geq 0$,*
- *$F(v + w) \leq F(v) + F(w)$ for all w at the same tangent space with v .*

Given a Finsler manifold (M, F) , the length of any rectifiable curve $\gamma : [a, b] \rightarrow M$ is given by the length functional

$$L[\gamma] = \int_a^b F(\gamma(t), \partial_t \gamma(t)) dt,$$

where $F(x, \cdot)$ is the Finsler norm on each tangent space $T_x M$.

The pair $(\mathcal{N}, \|\cdot\|)$ is a Finsler manifold, where \mathcal{N} denotes the set of normal matrices \mathcal{N} (of any size) and $\|\cdot\|$ denotes the operator norm.

Definition 3.3 (Matrix path curvature). *Given a piecewise- C^2 matrix path $\gamma : [0, 1] \rightarrow \mathcal{N}$, we define its curvature $\kappa[\gamma]$ to be*

$$\kappa[\gamma] := \frac{1}{\|\partial_t \gamma(t)\|} \left\| \partial_t \left(\frac{\partial_t \gamma(t)}{\|\partial_t \gamma(t)\|} \right) \right\|.$$

Definition 3.4 (Matrix flows). *Given $n \geq 1$, a mapping $\phi : \mathbb{R}_0^+ \times M_n \rightarrow M_n$, $(t, x) \mapsto x_t$ will be called a matrix flow in this document. If we have in addition that $\sigma(x_t) = \sigma(x_s)$ for every $t, s \geq 0$, we say that the matrix flow is isospectral.*

Definition 3.5 (interpolating path). *Given two matrices x and y in M_n and a matrix flow $\phi : \mathbb{I} \times M_n \rightarrow M_n$ such that $\phi_0(x) = x$ and $\phi_1(x) = y$, we say that the corresponding path $\{x_t\}_{t \in \mathbb{I}} := \{\phi_t(x)\}_{t \in \mathbb{I}} \subseteq M_n$ is a solvent path for the interpolation problem $x \rightsquigarrow y$.*

Definition 3.6 (\otimes operation). *Given two matrix paths $X, Y \in C([0, 1], M_n)$ we write $X \otimes Y$ to denote the concatenation of X and Y , which is the matrix path defined in terms of X and Y by the expression,*

$$X \otimes Y_s := \begin{cases} X_{2s}, & 0 \leq s \leq \frac{1}{2}, \\ Y_{2s-1}, & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Definition 3.7. ($\ell_{\|\cdot\|}$) *Given a matrix path $\{x_t\}_{t \in \mathbb{I}}$ in M_n we will write $\ell_{\|\cdot\|}(x_t)$ to denote the length of $\{x_t\}_{t \in \mathbb{I}}$ with respect to the operator norm which is defined by the expression*

$$\ell_{\|\cdot\|}(x_t) := \sup \sum_{k=0}^{m-1} \|x_{t_{k+1}} - x_{t_k}\|,$$

where the supremum is taken over all partitions of \mathbb{I} as $0 = t_0 < \dots < t_m = b$. If the function $x \in C(\mathbb{I}, M_n)$ is a piecewise C^1 function, then

$$\ell_{\|\cdot\|}(x_t) = \int_{\mathbb{I}} \|\partial_t x_t\| dt.$$

Definition 3.8. ($\|\cdot\|$ -flatness) *A set \mathcal{S} of M_n is said to be $\|\cdot\|$ -flat if any two points $x, y \in \mathcal{S}$ can be connected by a path $\{x_t\}_{t \in \mathbb{I}} \subseteq \mathcal{S}$ such that $\ell_{\|\cdot\|}(x_t) = \|x - y\|$.*

Definition 3.9 (Toroidal matrix link). *Given any two normal contractions x, y in M_n , a toroidal matrix link is any piecewise analytic normal path $x_t := \mathbb{K}[T_t(\mathbb{I}(x))]$ induced by a locally normal piecewise analytic matrix flow $T : \mathbb{I} \times M_N \rightarrow M_N$ with $N \geq n$, together with a locally normal compression $\mathbb{K} : M_N \rightarrow M_n$ with relative lifting map $\mathbb{I} : M_n \rightarrow M_N$, which satisfy the interpolating conditions $\mathbb{K}[T_0(\mathbb{I}(x))] = x$ and $\mathbb{K}[T_1(\mathbb{I}(x))] = y$ together with the constraints $\|\mathbb{K}[T_t(\mathbb{I}(x))]\| \leq 1$ for each $t \in \mathbb{I}$.*

Remark 3.1. *In the particular case where $[\mathbb{K}(T_t(\mathbb{I}(x))), \mathbb{K}(T_t(\mathbb{I}(y)))] = 0$ for each $t \in \mathbb{I}$, whenever $[x, y] = 0$, we call T a toral matrix link.*

Remark 3.2. *The curved nature of the matrix varieties (as Finsler submanifolds of \mathcal{N}) whose local connectivity is studied in this document, induces an obstruction to local connectivity via entirely flat toroidal matrix links in general. The toroidal matrix links $\mathbf{T} \subset C([0, 1], \mathcal{N})$ we have used to solve the connectivity problems which motivated this study satisfy the constraint*

$$0 \leq \kappa[T] \leq \frac{2}{\ell_{\|\cdot\|}(T)}, \quad \forall T \in \mathbf{T}.$$

3.2. Embedded matrix flows in solid tori. Given some fixed but arbitrary $W \in \mathbb{U}(n)$, using the operation $\text{diag}_n : M_n \rightarrow M_n$ one can define the mapping $\mathcal{D} : \mathbb{U}(n) \times M_n \rightarrow \mathbb{D}^2$, determined by the expression.

$$(3.4) \quad \mathbb{U}(n) \times M_n \rightarrow \mathbb{D}^2$$

$$(3.5) \quad (W, x) \mapsto \mathcal{D}_{\mathbb{T}}[W](x)$$

$$(3.6) \quad (W, x) \mapsto \{(\text{diag}_n [WxW^*])_{k,k}\}_{1 \leq k \leq n}$$

It is clear that $\text{diag}[\mathcal{D}_{\mathbb{T}}[W](x)] = \text{diag}_n [WxW^*]$ and that $\text{diag}[\mathcal{D}_{\mathbb{T}}[\mathbb{1}_n](x)] = \text{diag}_n [x]$, because of this when $W = \mathbb{1}_n$ we will write $\mathcal{D}(x)$ instead of $\mathcal{D}_{\mathbb{T}}[\mathbb{1}_n](x)$.

Given a matrix flow $\mathbb{I} \times \mathcal{N}(n)(\mathbb{D}^2) \rightarrow \mathcal{N}(n)(\mathbb{D}^2)$, $(t, x) \mapsto X_t(x)$, one can identify X with the set of flow lines in $\mathbb{D}^2 \times \mathbb{T}^1$ determined by $\{(\mathcal{D}(X_t(x)), e^{2\pi i t})\}_{t \in \mathbb{I}}$. The geometric picture determined by the mapping cylinder $\mathcal{N}(n)(\mathbb{D}^2) \times \mathbb{I} \rightarrow \mathbb{D}^2 \times \mathbb{T}^1$, $(x, t) \mapsto (\mathcal{D}(X_t(x)), e^{2\pi i t})$ will be called the embedded matrix mapping cylinder relative to the flow X . We can think of the embedded matrix mapping cylinder in topological terms as a deformation described by the expression \mathcal{D}_{X, Z_2} , which is defined as

$$(3.7) \quad \mathcal{D}_{X, Z_2}[Z_1 \times \mathbb{I}] := \frac{(Z_1 \times \mathbb{I}) \sqcup Z_2}{Z_1 \rightsquigarrow_{X_1} Z_2},$$

where Z_1 and Z_2 are some prescribed matrix varieties such that $x \in Z_1$ and $X_1(x) \in Z_2$.

Example 3.4 (Graphical example in M_3). *Let us set $\hat{u}_3 := e^{\frac{2\pi i}{3} f(\mathbf{N}_3)}$ where $f \in C(\mathbb{I}, \mathbb{I})$. For some prescribed $W_3 \in \mathbb{U}(3)$, we can obtain a graphical example of a particular geometric picture of the computation of the embedded matrix mapping cylinder relative to the interpolating flow \mathbf{U} which solves the problem $\hat{u}_3 \rightsquigarrow W_3 \hat{u}_3 W_3^*$.*

Let us set

$$\begin{aligned} Z_1 &:= \{z \in \mathbb{U}(3) | [\hat{u}_3, z] = 0\}, \\ Z_2 &:= \{z \in \mathbb{U}(3) | [W_3 \hat{u}_3 W_3^*, z] = 0\}. \end{aligned}$$

Using projective methods, we can trace specific flow lines along the matrix flows corresponding to the dynamical deformation $\mathcal{D}_{\mathbf{U}, Z_2}[Z_1 \times \mathbb{I}]$, which solve the interpolation problem $\hat{u}_3 \rightsquigarrow W_3 \hat{u}_3 W_3^$.*

A particular (approximate) geometric picture of the matrix deformation induced by the toral matrix link $\{\mathbf{U}_t\}_{t \in \mathbb{I}}$ in M_3 , projected in $\mathbb{D}^2 \times \mathbb{T}^1$ for each $t \in \mathbb{I}$ via $\mathcal{D}_{\mathbb{T}}(\mathbf{U}_t)$ is presented in figures F.1-F.3.

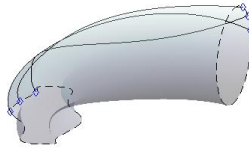


FIGURE 1. Projected matrix mapping cylinder corresponding to the path $\mathbf{U}_{[0, \frac{1}{2}]}(\hat{u}_3)$ in M_3 .

Alternative methods to trace particular flow lines on mapping cylinders can be obtained using matrix homotopies, this can be done using similar methods to the ones implemented in [7].

3.3. Environment algebras.

Definition 3.10 (Environment algebra (of a matrix algebra)). *Given a matrix algebra $A \subseteq M_n$, a universal C^* -algebra $\mathcal{E}_A := C_1^* \langle x_1, \dots, x_m | \mathcal{R}(x_1, \dots, x_m) \rangle$ for*

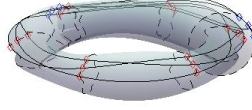


FIGURE 2. Projected matrix mapping cylinder corresponding to the path $\mathbf{U}_{\mathbb{I}}(\hat{u}_3)$ in M_3 .

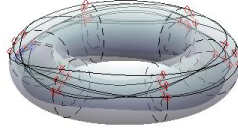


FIGURE 3. Embedded matrix mapping cylinder corresponding to the path $\mathbf{U}_{\mathbb{I}}(\hat{u}_3)$ in M_3 .

which there is a matrix representation $\mathcal{E}_A \rightarrow \mathbf{E}_A \subseteq M_n$ such that $A \subseteq \mathbf{E}_A$, will be called an **environment algebra** for A .

Let us consider the universal C^* -algebras $C(\mathbb{J})$, $C(\mathbb{T}^1)$, $C(\mathbb{T}^1) *_\mathbb{C} C(\mathbb{T}^1)$, $C_\delta(\mathbb{T}^2)$, $C_\delta(\mathbb{J} \times \mathbb{T}^1)$ and $C_\varepsilon^*\langle \mathbb{Z}/2 \times \mathbb{Z} \rangle$, defined in terms of generators and relations by the expressions.

$$C(\mathbb{J}) := C_1^* \langle u \mid h^* = h, \|h\| \leq 1 \rangle$$

$$C(\mathbb{T}^1) := C_1^* \langle u \mid uu^* = u^*u = 1 \rangle$$

$$C(\mathbb{T}^1) *_\mathbb{C} C(\mathbb{T}^1) := C_1^* \left\langle u, v \mid \begin{array}{l} uu^* = u^*u = 1, \\ vv^* = v^*v = 1 \end{array} \right\rangle$$

$$C_\delta(\mathbb{T}^2) := C_1^* \left\langle u, v \mid \begin{array}{l} uu^* = u^*u = 1, \\ vv^* = v^*v = 1, \\ \|uv - vu\| \leq \delta \end{array} \right\rangle$$

$$C_\delta(\mathbb{J} \times \mathbb{T}^1) := C_1^* \left\langle h, u \left| \begin{array}{l} h^* = h, \|h\| \leq 1 \\ uu^* = u^*u = 1, \\ \|hu - uh\| \leq \delta \end{array} \right. \right\rangle$$

$$C_\varepsilon^*(\mathbb{Z}/2 \times \mathbb{Z}) := C_1^* \left\langle u, v \left| \begin{array}{l} uu^* = u^*u = u^2 = 1, \\ vv^* = v^*v = 1, \\ \|uv - vu\| \leq \varepsilon \end{array} \right. \right\rangle.$$

Let us consider now a local matrix representation result that we will use later in the construction of particular representation schemes.

Lemma 3.1. *For every integer $n \geq 1$, there are $s_2, u_n, v_n \in \mathbb{U}(\mathcal{M}_\infty)$ such that the diagram*

$$\begin{array}{ccccc} C(\mathbb{T}^1)^{*2} & \longrightarrow & C^*(\langle \mathbb{Z}/n \rangle^{*2}) & \longrightarrow & C_n^*(u_n, v_n) \\ \downarrow & & & & \parallel \\ C^*(\mathbb{Z}/n * \mathbb{Z}/2) & \longrightarrow & C_n^*(s_2, v_n) & \xlongequal{\quad} & M_n \end{array}$$

commutes, where $s_2 \in \mathbb{H}(n)$, u_n and v_n are unitary elements in M_n .

Proof. Since we have that $C(\mathbb{T}^1)^{*2} \simeq C^*(\mathbb{F}_2) \simeq C^*(\mathbb{Z}^{*2})$, by universality of the C^* -representations

$$\begin{aligned} C^*(\mathbb{Z}^{*2}) &\simeq C^* \left\langle u, v \left| \begin{array}{l} uu^* = u^*u = \mathbb{1}, \\ vv^* = v^*v = \mathbb{1} \end{array} \right. \right\rangle \\ C^*(\langle \mathbb{Z}/n \rangle^{*2}) &\simeq C^* \left\langle u, v \left| \begin{array}{l} uu^* = u^*u = \mathbb{1}, \\ vv^* = v^*v = \mathbb{1}, \\ u^n = v^n = \mathbb{1} \end{array} \right. \right\rangle \\ C^*(\mathbb{Z}/n * \mathbb{Z}/2) &\simeq C^* \left\langle u, v \left| \begin{array}{l} uu^* = u^*u = \mathbb{1}, \\ vv^* = v^*v = \mathbb{1}, \\ u^n = v^2 = \mathbb{1} \end{array} \right. \right\rangle, \end{aligned}$$

and by the structural properties of M_n , it is enough to find for any $n \in \mathbb{Z}^+$, up to unitary congruence in M_n , three unitaries $s_2, u_n, v_n \in \mathbb{U}(n)$ such that $C^*(s_2, v_n) = M_n = C^*(u_n, v_n)$ and $u_n^n = v_n^n = s_2^2 = \mathbb{1}_n$, this can be done by taking for any $n \in \mathbb{Z}^+$ the orthogonal projection $p := \text{diag}[1, 0, \dots, 0] \in \mathbb{H}(n)$ and the matrix $s_2 = \mathbb{1} - 2p \in \mathbb{H}(n)$, setting $u_n := \Omega_n$ and $v_n := \Sigma_n$ for $n \geq 2$ and $u_1 = v_1 = \mathbb{1}$ for $n = 1$, by functional calculus and direct computations it is easy to verify that $s_2, u_n, v_n \in \mathbb{U}(n)$ for every $n \in \mathbb{Z}^+$, and that $s_2 = s_2^*$, it is also easy to verify that the system of matrix units $\{e_{i,j,n}\}_{1 \leq i,j \leq n}$ and u_n can be expressed as words in $C^*(s_2, v_n)$ for every $n \in \mathbb{Z}^+$, it is also clear that $p = e_{1,1,n}$ and hence, s_2 can be written as linear combinations of words in $C^*(u_n, v_n)$, we will then have that $C^*(\mathbb{Z}/n * \mathbb{Z}/2) \twoheadrightarrow C^*(v_n, s_2)$ and $C^*(\langle \mathbb{Z}/n \rangle^{*2}) \twoheadrightarrow C^*(u_n, v_n)$ by the universal properties of $C^*(\mathbb{Z}/2 * \mathbb{Z}/n)$ and $C^*(\langle \mathbb{Z}/n \rangle^{*2})$ respectively, since it can be easily verified that

$$u_n^n = v_n^n = s_2^2 = \mathbb{1}_n,$$

from these facts and the universal property of $C(\mathbb{T}^1)^{*2} \simeq C^*(\mathbb{F}_2) \simeq C^*(\mathbb{Z}^{*2})$, the result follows. \square

Remark 3.3. *It can be seen that for any matrix C^* -subalgebra $A \subseteq M_n$, there is $\delta > 0$ such that both $C(\mathbb{T}^1) *_{\mathbb{C}} C(\mathbb{T}^1)$ and $C_\delta(\mathbb{T}^2)$ are environment algebras of A . It can also be seen that for any abelian C^* -subalgebra $D \subseteq M_n$, $C(\mathbb{T}^1)$ is an environment algebra of D .*

4. LOCAL MATRIX CONNECTIVITY

4.1. Topologically controlled linear algebra and Soft Tori.

Definition 4.1 (Controlled sets of matrix functions). *Given $\delta > 0$, a function $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}_0^+$, a finite set of functions $F \subseteq C(\mathbb{T}^1, \mathbb{D}^2)$ and two unitary matrices $u, v \in M_n$ such that $\|uv - vu\| \leq \delta$, we say that the set F is δ -controlled by $\text{Ad}[v]$ if the diagram,*

$$\begin{array}{ccccc}
 C^*(u, v) & \longleftarrow & C^*(u) & \xleftarrow{i} & \{u\} \\
 & \swarrow \text{Ad}[v] & \downarrow & \searrow \text{Ad}[v] & \searrow f \\
 & & C^*(vuv^*) & \xleftarrow{i} & \{vuv^*\} \xrightarrow{f} \mathcal{N}(n)(\mathbb{D}^2)
 \end{array}$$

$\approx_{\varepsilon(\delta)}$

commutes up to an error $\varepsilon(\delta)$ for each $f \in F$.

Remark 4.1. *The C^* -homomorphism $C_\delta(\mathbb{T}^2) \rightarrow C^*(u, v)$ allows us to see that the Soft Torus $C_\delta(\mathbb{T}^2)$ provides an environment algebra for any δ -controlled set of matrix functions.*

Lemma 4.1 (Existence of isospectral approximants). *Given $\varepsilon > 0$ there is $\delta > 0$ such that, for any 2 families of N pairwise commuting normal matrices x_1, \dots, x_N and y_1, \dots, y_N which satisfy the constraints $\|x_j - y_j\| \leq \delta$ for each $1 \leq j \leq N$, there is a C^* -homomorphism Ψ such that $\sigma(\Psi(x_j)) = \sigma(x_j)$, $[\Psi(x_j), y_j] = 0$ and $\max\{\|\Psi(x_j) - y_j\|, \|\Psi(x_j) - x_j\|\} \leq \varepsilon$, for each $1 \leq j \leq N$.*

Proof. By changing basis if necessary, we can assume that y_1, \dots, y_N are diagonal matrices. From T.2.1 we will have that there is a permutation τ of the index set $\{1, \dots, n\}$ such that for each $1 \leq k \leq n$ we have that

$$\begin{aligned}
 |\Lambda^{(k)}(x_j) - \Lambda^{(\tau(k))}(y_j)| &\leq \|\Lambda^{(k)}(x_1, \dots, x_N) - \Lambda^{(\tau(k))}(y_1, \dots, y_N)\| \\
 (4.1) \qquad \qquad \qquad &\leq e_{N,0} \|\text{Cliff}(x_1 - y_1, \dots, x_N - y_N)\|.
 \end{aligned}$$

Using 2.6 and as a consequence of 4.1 we can find a permutation matrix $\mathcal{T} \in \mathbb{U}(n)$ such that

$$\begin{aligned}
 \|\mathcal{T}^* \text{diag} [\Lambda(x_j)] \mathcal{T} - \text{diag} [\Lambda(y_j)]\| &\leq e_{N,0} \|\text{Cliff}(x_1 - y_1, \dots, x_N - y_N)\| \\
 (4.2) \qquad \qquad \qquad &\leq e_{N,0} N \delta, \quad 1 \leq j \leq N.
 \end{aligned}$$

Let us set $c_N := e_{N,0} N$. For the matrices x_1, \dots, x_N there is a unitary joint diagonalizer $W \in M_n$ such that $W \text{diag} [\Lambda(x_j)] W^* = x_j$, $1 \leq j \leq N$,

$$\begin{aligned}
 \|W \text{diag} [\Lambda(x_j)] W^* - \mathcal{T}^* \text{diag} [\Lambda(x_j)] \mathcal{T}\| &\leq \|W \text{diag} [\Lambda(x_j)] W^* - y_j\| \\
 &\quad + \|y_j - \mathcal{T}^* \text{diag} [\Lambda(x_j)] \mathcal{T}\| \\
 (4.3) \qquad \qquad \qquad &\leq (1 + c_N) \|x_j - y_j\| \leq (1 + c_N) \delta.
 \end{aligned}$$

If we set $V := W\mathcal{T}$ and $\varepsilon = (1 + c_N)\delta$, we will have that by 4.2 and 4.3 the inner C^* -automorphism $\Psi := \text{Ad}[V^*]$ satisfies the constraints in the statement of this lemma, and we are done. \square

Remark 4.2. The C^* -automorphism Ψ from L.4.1 is called an isospectral approximant for the two N -tuples x_1, \dots, x_N and y_1, \dots, y_N . If $\Psi := \text{Ad}[W^*]$ for some $W \in \mathbb{U}(n)$, then we will have that its inverse Ψ^\dagger will be given by the expression $\Psi^\dagger = \text{Ad}[W]$.

Remark 4.3. The constant c_N in the proof of L.4.1 depends only on the number N of matrices in each family. It does not depend on the matrix size.

4.2. Local piecewise analytic connectivity. In this section we will present some piecewise analytic local connectivity results in matrix representations of the form $C_\varepsilon(\mathbb{T}^2) \rightarrow M_n \leftarrow C(\mathbb{T}^N)$ and $C_\varepsilon(\mathbb{J} \times \mathbb{T}^1) \rightarrow M_n \leftarrow C(\mathbb{J}^N)$.

Theorem 4.1 (Local normal toral connectivity). *Given $\varepsilon > 0$ and any $n \in \mathbb{Z}^+$, there is $\delta > 0$ such that, for any $2N$ normal contractions x_1, \dots, x_N and y_1, \dots, y_N in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, & 1 \leq j, k \leq N, \\ \|x_j - y_j\| \leq \delta, & 1 \leq j \leq N, \end{cases}$$

there exist N toral matrix links X^1, \dots, X^N in M_n , which solve the problems

$$x_j \rightsquigarrow y_j, \quad 1 \leq j \leq N,$$

and satisfy the constraints

$$\left\{ \|X_t^j(x_j) - y_j\| \leq \varepsilon, \right.$$

for each $1 \leq j, k \leq N$ and each $t \in \mathbb{I}$. Moreover, $\ell_{\|\cdot\|}(X_t^j(x_j)) \leq \varepsilon, \quad 1 \leq j \leq N$.

Proof. By L.2.1, L.2.2 and L.4.1 we will have that given $\varepsilon > 0$, there are $0 < \delta \leq \nu \leq \varepsilon/2$ and an isospectral approximant $\Psi := \text{Ad}[W^*]$ (with $W \in \mathbb{U}(n)$) for x_1, \dots, x_N and y_1, \dots, y_N such that, $\max\{\|x_j - \Psi(x_j)\|, \|y_j - \Psi(x_j)\|\} \leq \nu$ and $[\Psi(x_j), y_j] = 0$ for each $1 \leq j \leq N$, we will also have that there is a unitary path $\mathcal{W} \in C(\mathbb{I}, M_n)$ which is defined by the expression $\mathcal{W}_t := e^{-itH_W}$ for each $t \in \mathbb{I}$, where $H_W \in M_n$ is a hermitian matrix such that $e^{iH_W} = W$ and $\|[H_W, x_j]\| \leq \varepsilon/2$ for each $1 \leq j \leq N$, and that is defined by $H_W := h(W)$, for some function $h : \Omega_{d,s}^\alpha \rightarrow [-1, 1]$, and where $\sigma(W) \subset \Omega_{d,s}^\alpha := \{e^{i(\pi t + \alpha)} - 1 + s < t < 1 - s\} \subset \mathbb{T}^1$, with $s, \alpha \in \mathbb{R}$ chosen in such a way that $\mathbb{T}^1 \setminus \Omega_{d,s}^\alpha$ contains an arc of length d (with $d \geq 2\pi/n$). Moreover, we can choose δ and ν in such a way that the path \mathcal{W} satisfies the inequalities $\|[\mathcal{W}_t, \Psi(x_j)]\| \leq \varepsilon/2$ for each $t \in [0, 1]$ and each $1 \leq j \leq N$.

It can be seen that the paths $\check{X}_t^j := \text{Ad}[\mathcal{W}_t](x_j)$ will solve the problem $x_j \rightsquigarrow_{\varepsilon/2} \Psi(x_j)$ for each $1 \leq j \leq N$. Let us set $\bar{X}_t^j := (1-t)\Psi(x_j) + ty_j$, we can now construct N toroidal matrix links of the form $X^j := \check{X}^j \circledast \bar{X}^j$ which solve the problems $x_j \rightsquigarrow y_j$, locally preserve normality and commutativity and satisfy the $\|\cdot\|$ -distance constraints

$$\begin{aligned} \|X_t^j - y_j\| &\leq \|X_t^j - \Psi(x_j)\| + \|y_j - \Psi(x_j)\| \\ &\leq \frac{\varepsilon}{2} + \nu \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

together with the $\|\cdot\|$ -length constraints

$$(4.4) \quad \ell_{\|\cdot\|}(X_t^j) \leq \ell_{\|\cdot\|}(\check{X}_t^j) + \|\Psi(x_j) - y_j\|$$

$$(4.5) \quad = \int_{\mathbb{I}} \|\partial_t \text{Ad}[\mathcal{W}_t](x_j)\| dt + \|\Psi(x_j) - y_j\|$$

$$(4.6) \quad = \|[H_W, \Psi(x_j)]\| + \|\Psi(x_j) - y_j\|$$

$$(4.7) \quad \leq \frac{\varepsilon}{2} + \nu \leq \varepsilon,$$

which hold whenever $\|x_j - y_j\| \leq \delta$, $1 \leq j \leq N$, and we are done. \square

Remark 4.4. *It can be noticed that the solvent matrix links X^1, \dots, X^N whose existence is stated in T.4.1 are factored in the form $X^j = \check{X}^j \otimes \bar{X}^j$, we call \check{X}^j and \bar{X}^j the **curved** and **flat** factors of X^j respectively.*

We will derive now, some corollaries of the proof of T.4.1.

Corollary 4.1 (Local hermitian toral connectivity). *Given $\varepsilon > 0$ and any integer $n \geq 1$, there is $\delta > 0$ such that, for any $2N$ hermitian contractions x_1, \dots, x_N and y_1, \dots, y_N in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, & 1 \leq j, k \leq N, \\ \|x_j - y_j\| \leq \delta, & 1 \leq j \leq N, \end{cases}$$

there exist N toral matrix links X^1, \dots, X^N in M_n , which solve the problems

$$x_j \rightsquigarrow y_j, \quad 1 \leq j \leq N,$$

and satisfy the constraints

$$\begin{cases} X_t^j(x_j) = (X_t^j(x_j))^*, \\ \|X_t^j(x_j) - y_j\| \leq \varepsilon, \end{cases}$$

for each $1 \leq j, k \leq N$ and each $t \in \mathbb{I}$. Moreover, $\ell_{\|\cdot\|}(X_t^j(x_j)) \leq \varepsilon$, $1 \leq j \leq N$.

Proof. Since for any $\alpha \in \mathbb{R}$, any pair of hermitian matrices $x, y \in \mathbb{H}(n)$ and any partial unitary $z \in \mathbb{P}\mathbb{U}(n)$, we have that $x + \alpha(y - x)$ and zxz^* are also in $\mathbb{H}(n)$, the result follows as a consequence of L.4.1 and T.4.1. \square

Corollary 4.2 (Local unitary toral connectivity). *Given any $\varepsilon \geq 0$ and any integer $n \geq 1$, there is $\delta \geq 0$ such that given any $2N$ unitary matrices $U_1, \dots, U_N, V_1, \dots, V_N$ in M_n which satisfy the relations*

$$\begin{cases} [U_j, U_k] = [V_j, V_k] = 0, \\ \|U_k - V_k\| \leq \delta, \end{cases}$$

for each $1 \leq j, k \leq N$, there are toral matrix links u^1, \dots, u^N in M_n which solve the interpolation problems

$$U_k \rightsquigarrow V_k, \quad 1 \leq k \leq N,$$

and also satisfy the relations

$$\begin{cases} (u_t^j)^* u_t^j = u_t^j (u_t^j)^* = \mathbb{1}_n, \\ \|u_t^j - V_j\| \leq \varepsilon, \end{cases}$$

for each $t \in \mathbb{I}$ and each $1 \leq j, k \leq N$. Moreover, $\ell_{\|\cdot\|}(u_t^j) \leq \varepsilon$, $1 \leq j \leq N$.

Proof. Since for any C^* -automorphisms Ψ we have that $\Psi(\mathbb{U}(n)) \subseteq \mathbb{U}(n)$, and since any two commuting unitaries U and V can be connected by a flat unitary path $\bar{U}_t := Ue^{t \ln(U^*V)}$, for $0 \leq t \leq 1$. We will have that the result can be derived using a similar argument to the one implemented in the proof of T.4.1. \square

4.2.1. Lifted local piecewise analytic connectivity. Let us denote by κ the matrix compression $M_{2n} \rightarrow M_n$ defined by the mapping

$$\kappa : M_{2n} \rightarrow M_n, \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x_{11}.$$

Let us write $\iota_2 : M_n \rightarrow M_{2n}$ to denote the C^* -homomorphism defined by the expression $\iota_2(x) := x \oplus x = \mathbb{1}_2 \otimes x$.

Definition 4.2 (Standard dilations). *Given a C^* -automorphism $\Psi := \text{Ad}[W]$ (with $W \in \mathbb{U}(n)$) in M_n , we will denote by $\Psi^{[s]}$ the C^* -automorphism in M_{2n} defined by the expression $\Psi^{[s]} := \text{Ad}[\mathbb{1}_2 \otimes W] = \text{Ad}[W \oplus W]$. We call $\Psi^{[s]}$ a standard dilation of Ψ .*

Definition 4.3 ($\mathbb{Z}/2$ -dilations). *Given a C^* -automorphism $\Psi := \text{Ad}[W]$ (with $W \in \mathbb{U}(n)$) in M_n , we will denote by $\Psi^{[2]}$ the C^* -automorphism in M_{2n} defined by the expression $\Psi^{[2]} := \text{Ad}[(\Sigma_2 \otimes \mathbb{1}_n)(W^* \oplus W)]$. We call $\Psi^{[2]}$ a $\mathbb{Z}/2$ -dilation of Ψ .*

Remark 4.5. *It can be seen that $\kappa(\iota_2(x)) = x$ for any $x \in M_{2n}$, it can also be seen that $\kappa(\Psi^{[2]}(\iota_2(x))) = \kappa(\Psi^{[s]}(\iota_2(x)))$.*

Theorem 4.2 (Lifted local toral connectivity). *Given $\varepsilon > 0$, there is $\delta > 0$ such that, for any $2N$ normal contractions x_1, \dots, x_N and y_1, \dots, y_N in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, & 1 \leq j, k \leq N, \\ \|x_j - y_j\| \leq \delta, & 1 \leq j \leq N, \end{cases}$$

there is a C^ -homomorphism $\Phi : M_n \rightarrow M_{2n}$ and N toral matrix links X^1, \dots, X^N in $C(\mathbb{T}, M_{2n})$, which solve the problems*

$$\Phi(x_j) \rightsquigarrow y_j \oplus y_j, \quad 1 \leq j \leq N,$$

and satisfy the constraints

$$\begin{cases} \kappa(\Phi(x_j)) = x_j, \\ \|\Phi(x_j) - x_j \oplus x_j\| \leq \varepsilon, \\ \|X_t^j - y_j \oplus y_j\| \leq \varepsilon, \end{cases}$$

for each $1 \leq j, k \leq N$ and each $t \in \mathbb{T}$. Moreover, $\ell_{\|\cdot\|}(X_t^j) \leq \varepsilon$, $1 \leq j \leq N$.

Proof. By L.4.1 we will have that given $\varepsilon > 0$, there are $0 < \delta \leq \nu = \frac{\varepsilon}{2\pi}$ and an isospectral approximant $\Psi := \text{Ad}[W^*]$ (with $W \in \mathbb{U}(n)$) for x_1, \dots, x_N and y_1, \dots, y_N such that, $\max\{\|x_j - \Psi(x_j)\|, \|y_j - \Psi(x_j)\|\} \leq \nu$. By setting $\Phi := (\Psi^\dagger)^{[2]} \circ \iota_2 \circ \Psi$, by D.4.3, D.4.2 and R.4.5 it can be seen that $\Phi : M_n \rightarrow M_{2n}$ is a C^* -homomorphism such that $\|\Phi(x_j) - \iota_2(x_j)\| = \|\Phi(x_j) - x_j \oplus x_j\| \leq \varepsilon$, for each $1 \leq j \leq N$.

Since $(\Psi^\dagger)^{[2]} := \text{Ad}[\hat{W}_s]$ with $\hat{W}_s := (\Sigma_2 \otimes \mathbb{1}_n)(W^* \oplus W)$ and since $\hat{W}_s \in \mathbb{U}(2n) \cap \mathbb{H}(2n)$, we will have that \hat{W}_s can be represented as $\hat{W}_s = e^{i\frac{\pi}{2}(\hat{W}_s - \mathbb{1}_{2n})}$ for any $n \geq 1$. If we set $\tilde{X}_j := \Psi^{[s]}(\iota_2(x_j))$, $1 \leq j \leq N$, we also have that there is a unitary

path $\{\mathcal{W}_t\}_{t \in \mathbb{I}} \subset M_{2n}$ with $\mathcal{W}_t := e^{i\frac{\pi(1-t)}{2}(\hat{W}_s - \mathbb{1}_{2n})}$, which satisfies the conditions $\mathcal{W}_0 = \hat{W}_s$, $\mathcal{W}_1 = \mathbb{1}_{2n}$, together with the normed estimates

$$\begin{aligned} \|\mathcal{W}_t \tilde{X}_j - \tilde{X}_j \mathcal{W}_t\| &= |\cos(\pi t/2)| \|\hat{W}_s \tilde{X}_j - \tilde{X}_j \hat{W}_s\| \\ &\leq \|\hat{W}_s \tilde{X}_j - \tilde{X}_j \hat{W}_s\| \leq \nu, \end{aligned}$$

for each $1 \leq j \leq N$ and each $0 \leq t \leq 1$. Moreover, for each $1 \leq j \leq N$ we have that the paths $\check{X}_t^j := \text{Ad}[\mathcal{W}_t](\tilde{X}_j)$ satisfy the normed estimates

$$\begin{aligned} \ell_{\|\cdot\|}(\check{X}_t^j) &= \int_{\mathbb{I}} \|\partial_t \text{Ad}[\mathcal{W}_t](\tilde{X}_j)\| dt, \\ &= \frac{\pi}{2} \|\hat{W}_s \tilde{X}_j - \tilde{X}_j \hat{W}_s\| \leq \nu. \end{aligned}$$

For each $1 \leq j \leq N$, we can now use the flat paths $\bar{X}_t^j := (1-t)\tilde{X}_j + t\iota_2(y_j)$ together with the previously described curved paths \check{X}_t^j to construct the solvent toral matrix links $X^1, \dots, X^N \in C([0, 1], M_{2n})$ we are looking for, and which can be defined by $X^j := \check{X}^j \circ \bar{X}^j$ for each $1 \leq j \leq N$, and we are done. \square

Remark 4.6. *It can be seen that by using the technique implemented in the proof of T.4.2 one can obtain lifted versions of C.4.2 and C.4.1.*

Remark 4.7. *As a consequence of T.4.2 we can derive simple detection methods to identify families of pairwise commuting matrices in M_n that can be connected uniformly via piecewise analytic toral matrix links. The existence of these detection methods raises some interesting questions for further studies.*

Remark 4.8. *We can interpret T.4.2 as an existence theorem of solutions to lifted connectivity problems defined on matrix representations of the form*

$$\begin{array}{ccccccc} & & C_\varepsilon^*(\mathbb{Z}/2 \times \mathbb{Z}) & \longrightarrow & C^*(\hat{U}_s, \hat{V}) & \longrightarrow & M_{2n} \\ & \nearrow & & & \uparrow & & \downarrow \\ C^*(\mathbb{F}_2) & \longrightarrow & C_\delta(\mathbb{T}^2) & \longrightarrow & C^*(U, V) & \longrightarrow & M_n \end{array}$$

with $\hat{U}_s = (\Sigma_2 \otimes \mathbb{1}_n)(U^* \oplus U)$ and $\hat{V} = V \oplus V$.

Some further applications of T.4.2 to approximation of matrix words and norm behavior will be presented in [33].

4.2.2. Matrix Klein Bottles: Local matrix deformations and special symmetries. Using T.4.2 we can solve all connectivity problems (together with their softened versions) in M_n that can be reduced to connectivity problems of the form $x \rightsquigarrow_\varepsilon x^*$ in $\mathcal{N}(n)(\mathbb{D}^2)$, with $x^* = T x T$ and $T^2 = \mathbb{1}_n$.

Remark 4.9. *For each $\varepsilon \in [0, 2]$, we can use the previously described symmetries and $\mathcal{D}_\mathbb{T}$ to interpret $\bigcup_{x \in M_n} \{x \rightsquigarrow_\varepsilon x^*\}$ as matrix analogies of the Klein bottle.*

By a *softened matrix Klein bottle* we mean that the symmetries are softened, in particular we can consider the connectivity problems $x \rightsquigarrow_\varepsilon x^*$ and $y \rightsquigarrow_\varepsilon y^*$ in $\mathcal{N}(n)(\mathbb{D}^2)$ subject to the normed constraints $\|xy - yx\| \leq \delta$, $\|x^* - T x T\|, \|x T - T y\| \leq \delta$ and $T^2 = \mathbb{1}_n$. The details regarding to the solvability of these local connectivity problems will be addressed in future communications.

4.3. C^0 uniform local connectivity of pairs of unitaries and piecewise analytic approximants. The technique presented in this section can be used to solve local connectivity problems in matrix representations of the form $C_\epsilon(\mathbb{T}^2) \rightarrow M_n \leftarrow C(\mathbb{T}^2)$ uniformly via C^0 -unitary paths.

Suppose U_t and V_t are unitary matrices in $\mathbf{M}_n(\mathbb{C})$ for $t = 0$ and $t = 1$ and we define

$$(4.8) \quad U_t = U_0 e^{t \ln(U_0^* U_1)}$$

and

$$(4.9) \quad V_t = V_0 e^{t \ln(V_0^* V_1)}.$$

For $t = 0$ or $t = 1$ the C^* -algebra generated by U_t and V_t is abelian, so select a MASA $C_t \cong \mathbb{C}^n$ in each case. Let

$$A(C_0, C_1) = \{X \in C([0, 1], \mathbf{M}_n(\mathbb{C})) \mid X(0) \in C_0 \text{ and } X(1) \in C_1\}.$$

Lemma 4.2. *The C^* -algebra $A(C_0, C_1)$ has stable rank one.*

Proof. Starting with X continuous with $X(t)$ in C_t at the endpoints, we can adjust this by a small amount, leaving the endpoints in C_t , to get X piece-wise linear, with the endpoints of every linear segment having no spectral multiplicity and being invertible. Using Kato's theory of analytic paths, we can get a piece-wise continuous unitary U_t and piece-wise analytic scalar paths $\lambda_n(t)$ so that the new path $Y \approx X$ satisfies

$$Y(t) = U_t \begin{bmatrix} \lambda_1(t) & & \\ & \ddots & \\ & & \lambda_n(t) \end{bmatrix} U_t^*.$$

There may be finitely many places where $Y(t)$ is not invertible. These places will be in the interior of the segment so in an open interval where U_t is continuous. A small deformation of some of the λ_j will take the path through invertibles. We have not moved the endpoints in the second adjustment so the constructed element is in $A(C_0, C_1)$ and close to X . \square

Lemma 4.3. *The endpoint-restriction map $\rho : A(C_0, C_1) \rightarrow C_0 \oplus C_1$ induces an injection on K_0 .*

Proof. The kernel of ρ is $C([0, 1], \mathbf{M}_n(\mathbb{C}))$ which has trivial K_0 -group. So this result follows from the exactness of the usual six-term sequence in K -theory. \square

Lemma 4.4. *Given unitaries U and V in $A(C_0, C_1)$, with $\|[U, V]\| \nu_0$ as if D.2.2 (so the Bott index makes sense), $\text{Bott}(U, V)$ is the trivial element of $K_0(A(C_0, C_1))$.*

Proof. By the previous lemma, we need only calculate $\text{Bott}(\rho(U), \rho(V))$. These unitaries are in a commutative C^* -algebra so they have trivial Bott index. \square

Theorem 4.3. *Given $\epsilon > 0$, there exists $\delta > 0$ so that for all n , given unitary matrices U_0, U_1, V_0, V_1 in $\mathbf{M}_n(\mathbb{C})$ with $U_0 V_0 = V_0 U_0$, $U_1 V_1 = V_1 U_1$, $\|U_0 - U_1\| \leq \delta$ and $\|V_0 - V_1\| \leq \delta$, then there exists continuous paths U_t and V_t between the given pairs of unitaries with each U_t and V_t unitary, and with $U_t V_t = V_t U_t$, $\|U_t - U_0\| \leq \epsilon$ and $\|V_t - V_0\| \leq \epsilon$ for all t .*

Proof. The paths U_t and V_t defined in equations 4.8 and 4.9 will be almost commuting unitary elements of $A(C_0, C_1)$. By Lemma 4.2 we may apply [9, Theorem 8.1.1] regarding approximating in $A(C_0, C_1)$ by commuting unitaries. Lemma 4.4 tells us there is no invariant to worry about, so we can find A_t and B_t close of U_t and V_t that are commuting continuous paths of unitaries with A_t and B_t in C_t for $t = 0, 1$. The unitary elements in the commutative C_t are locally connected, so we can find a short path from U_0 and V_0 to A_0 and B_0 , and likewise at the other end. Concatenating, we get a paths of commuting unitary matrices from U_0 and V_0 to U_1 and V_1 so that at every point we are close to some pair (U_t, V_t) . These then are all close to U_0 and V_0 . \square

By combining T.4.2, C.4.2 and T.4.3 it can be seen that.

Remark 4.10 (Piecewise analytic approximants of C^0 interpolants). *Given $\epsilon > 0$, there exists $\delta > 0$ so that for all n , given unitary matrices U_0, U_1, V_0, V_1 in $M_n(\mathbb{C})$ with $U_0V_0 = V_0U_0, U_1V_1 = V_1U_1, \|U_0 - U_1\| \leq \delta$ and $\|V_0 - V_1\| \leq \delta$, there exist continuous (interpolants) paths U_t and V_t in M_{2n} which solve the problems $U_0 \oplus U_0 \rightsquigarrow U_1 \oplus U_1$ and $V_0 \oplus V_0 \rightsquigarrow V_1 \oplus V_1$ with each U_t and V_t unitary, and with $U_tV_t = V_tU_t, \|U_t - U_0 \oplus U_0\| \leq \epsilon$ and $\|V_t - V_0 \oplus V_0\| \leq \epsilon$ for all t . There are also a C^* -homomorphism $\Psi : M_n \rightarrow M_{2n}$ such that*

$$\max\{\|\Psi(U_0) - U_1 \oplus U_1\|, \|\Psi(U_0) - U_0 \oplus U_0\|, \|\Psi(V_0) - V_1 \oplus V_1\|, \|\Psi(V_0) - V_0 \oplus V_0\|\} \leq \epsilon,$$

and two piecewise analytic unitary pairwise commuting paths $\hat{U}, \hat{V} \in C([0, 1], M_{2n})$ which solve the problems $\Psi(U_0) \rightsquigarrow U_1 \oplus U_1, \Psi(V_0) \rightsquigarrow V_1 \oplus V_1$ with $\max\{\|\hat{U}_t - U_t\|, \|\hat{V}_t - V_t\|\} \leq \epsilon$ for each $0 \leq t \leq 1$. Moreover, $\ell_{\|\cdot\|}(\hat{U}_t) \leq \epsilon$ and $\ell_{\|\cdot\|}(\hat{V}_t) \leq \epsilon$.

4.4. Jointly compressible matrix sets. Given $0 < \delta \leq \epsilon$, we can now consider an alternative approach to the local connectivity problem involving two N -sets of pairwise commuting normal matrix contractions X_1, \dots, X_N and Y_1, \dots, Y_N such that $\|X_j - Y_j\| \leq \delta$ for each $1 \leq j \leq N$. The approach that we will consider in this section consists of considering the existence of a normal contraction \hat{X} such that $X_1, \dots, X_N \in C^*(\hat{X})$, and which also satisfies the constraint $\|\hat{X} - X_j\| \leq \epsilon$ for some $1 \leq j \leq N$. A matrix \hat{X} which satisfies the previous conditions will be called a **nearby generator** for X_1, \dots, X_N , it can be seen that for any $\delta \leq \nu \leq \epsilon$ one can find a flat analytic path $\bar{X} \in C([0, 1], M_\infty)$ that performs the deformation $X_j \rightsquigarrow_\nu \hat{X}$, where \hat{X} is a nearby generator for X_1, \dots, X_N .

Given any joint isospectral approximant Ψ with respect to the families normal contractions described in the previous paragraph, along the lines of the program that we have used to derive the connectivity results T.4.1 and T.4.2, we can use L.4.1 to find a C^* -automorphism which solves the extension problem described by the diagram,

$$(4.10) \quad \begin{array}{ccc} & & C^*(\hat{X}) \\ & \nearrow & \downarrow \hat{\Psi} \\ C^*(X_1, \dots, X_N) & \xrightarrow{\Psi} & C^*(Y_1, \dots, Y_N) \end{array}$$

and satisfies the relations $\Psi(X_j) = \hat{\Psi}(X_j)$ for each $1 \leq j \leq N$ together with the normed constraints

$$\max\{\|\hat{\Psi}(\hat{X}) - \hat{X}\|, \max_j\{\|\hat{\Psi}(X_j) - X_j\|, \|\hat{\Psi}(X_j) - Y_j\|\}\} \leq \varepsilon.$$

We refer to the C^* -automorphism $\hat{\Psi}$ in 4.10 as a **compression** of Ψ or a **compressive joint isospectral approximant (CJIA)** for the N -sets of normal contractions. Let us now consider a special type of inner C^* -automorphisms that can be described as follows.

Definition 4.4 (Uniformly compressible JIA). *Given $0 < \delta \leq \varepsilon$ and two N -sets of pairwise commuting normal contractions X_1, \dots, X_N and Y_1, \dots, Y_N in \mathcal{M}_∞ such that $\|X_j - Y_j\| \leq \delta$, $1 \leq j \leq N$, a joint isospectral approximant Ψ of the N -sets is said to be **uniformly compressible** if there are a nearby generator \hat{X} for X_1, \dots, X_N , a compression $\hat{\Psi} := \text{Ad}[W]$ (with $W \in \mathcal{U}(\mathcal{M}_\infty)$) of Ψ and a unitary $\hat{W} \in \hat{\Psi}(C^*(\hat{X}))$ such that $\|W - \hat{W}\| \leq \varepsilon$. We refer to the $2N$ normal contractions X_1, \dots, X_N and Y_1, \dots, Y_N for which there exists a uniformly compressible JIA (**UCJIA**) as **uniformly jointly compressible (UJC)**.*

Lemma 4.5 (Local connectivity of **UJC** matrix sets). *Given $\varepsilon > 0$, there is $\delta > 0$, such that for any two N -sets of **UJC** pairwise commuting normal contractions X_1, \dots, X_N and Y_1, \dots, Y_N in \mathcal{M}_∞ such that $\|X_j - Y_j\| \leq \delta$ for each $1 \leq j \leq N$, we will have that there are N toral matrix links $\mathbf{X}^1, \dots, \mathbf{X}^N \in C([0, 1], \mathcal{M}_\infty)$ that solve the interpolation problem $X_j \rightsquigarrow_\varepsilon Y_j$, for each $1 \leq j \leq N$.*

Proof. Since the N -sets of pairwise commuting normal contractions X_1, \dots, X_N and Y_1, \dots, Y_N are **UJC**, we have that given $0 < \delta \leq \nu \leq \varepsilon/2 < 1$, there are a normal contraction $\hat{X} \in \mathcal{M}_\infty$ which commutes with each X_j together with a **UCJIA** $\hat{\Psi} = \text{Ad}[W]$ for some $W \in \mathcal{U}(\mathcal{M}_\infty)$ and a unitary $\hat{W} \in \hat{\Psi}(C^*(\hat{X}))$ such that

$$(4.11) \quad \|\mathbb{1} - \hat{W}^*W\| = \|W - \hat{W}\| \leq \nu < 1.$$

Let us set $Z := \hat{W}^*W$, as a consequence of the inequality 4.11 we will have that there is a hermitian matrix $-\mathbb{1} \leq H_Z \leq \mathbb{1}$ in \mathcal{M}_∞ such that $e^{\pi i H_Z} = Z$. By using 4.11 again, it can be seen that we can now use the curved paths $\check{\mathbf{X}}^j := \text{Ad}[e^{-\pi i t H_Z}](X_j)$ to solve the problems $X_j \rightsquigarrow_{\varepsilon/2} \hat{\Psi}(X_j)$, and then we can solve the problems $\hat{\Psi}(X_j) \rightsquigarrow_\nu Y_j$ using the flat paths $\bar{\mathbf{X}}^j := (1 - t)\hat{\Psi}(X_j) + tY_j$. We can construct the solvent toral matrix links by setting $\mathbf{X}^j := \check{\mathbf{X}}^j \circledast \bar{\mathbf{X}}^j$ for each $1 \leq j \leq N$. This completes the proof. \square

5. HINTS AND FUTURE DIRECTIONS

The detection matrix representations of universal C^* -algebras that can be connected uniformly via piecewise analytic paths induces interesting problems which are topological/K-theoretical and computational in nature. Motivated by the C^0 -connectivity technique, we consider that the use of T.4.1, C.4.2 and T.4.2 and L.4.5 to study local matrix connectivity in C^* -representations of the form

$$C(\mathbb{T}^N) \begin{array}{c} \xrightarrow{\quad} M_{2n} \xleftarrow{\quad} M_n \xleftarrow{\quad} C_\varepsilon(\mathbb{T}^2) \\ \quad \quad \quad \nwarrow \quad \quad \nearrow \end{array}$$

will present interesting challenges and questions that will be the subject of future study. In particular we are interested in the application of T.4.2, C.4.2 and L.4.5

to the study of the question. Is $C^*\langle \mathbb{F}_2 \times \mathbb{F}_2 \rangle$ RFD? (This is equivalent to **Connes's embedding problem**.)

A better understanding of the geometric and approximate combinatorial nature of toroidal matrix links would provide a mutually beneficial interaction between matrix flows in the sense of Brockett [4] and Chu [8], topologically controlled linear algebra in the sense of Freedman and Press [14] and matrix geometric deformations in the sense of Loring [25]. This also may provide some novel generic numerical methods to study and compute normal matrix compressions, sparse representations and dimensionality reduction of large scale matrices. Using a similar approach we plan to use T.4.2 and L.4.5 to answer some questions in topologically controlled linear algebra in the sense of [14], raised by M. H. Freedman.

The construction and generalization of detection methods like the ones mentioned in the remark R.4.7 of theorem T.4.2 together with their implications on inverse spectral problems, will be the subject of future studies. In particular we will use toral matrix links to study the local deformation properties of matrix representations of the form $C_\varepsilon(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}/2 \rightarrow M_n$ (where α denotes the standard flip) via *softened matrix Klein bottles*. These problems are related to spectral decomposition problems with *spectral symmetry* in quantum theory and to *deformation theory* for C^* -algebras in the sense of Loring [25]. We will also use toroidal matrix links to study the local connectivity of some Soft group C^* -algebras in the sense of Farsi [13].

Some generalizations of T.4.2 and particular applications of L.4.5 to the study of matrix equations on words will also be the subject of future study. In particular, the combination of toroidal matrix links with some matrix lifting techniques along the same lines of the proof of T.4.2 combined with L.4.5, seem also promising on the solvability of some conjectures studied numerically on [28].

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